New exact Taylor's expansions and simple solutions to PDEs

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Abstract. We provide new *exact* Taylor's series with fixed coefficients and without the remainder. We demonstrate the usefulness of this contribution by using it to obtain very simple solutions to (non-linear) PDEs. We also apply the method to the portfolio model.

Key words: exact Taylor's series, remainder, PDEs, heat equation, reaction-convection-diffusion equation, portfolio.

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1 Introduction

In this paper, we provide an *exact* Taylor's expansion (Taylor (1715)) with *constant* coefficients and without the remainder. In doing so, we provide an explicit (closed) form of the remainder. Needless to say, this pioneering contribution is extremely useful in many applications, such as the areas of optimization, integration, and partial differential equations PDEs, since it transforms any arbitrary function to an exact explicit quasi-linear form. Consequently, this method will simplify the solutions to cumbersome integrals, PDEs and optimization problems (see Alghalith (2008) for potential applications). Consequently, this contribution can be applied to many areas in operations research, such as optimization, stochastic models, and statistics.

We apply our approach to the heat equation and the reaction-convection-diffusion equation. In doing so, we use it to transform a (non-linear) PDE to a non-differential equation or an ordinary differential equation ODE. We also apply our method to the dynamic portfolio model in finance. In so doing, unlike previous literature, we derive an explicit solution to the investor's optimal portfolio when the utility function is unknown.

2 The new expansions

Theorem: A sufficiently differentiable function with a compact support $f(.)$ is given by

(i) $f(x) = a_1 + a_2 x + a_2 (x + a_3) \ln (x + a_3)$,

(i) $f(x, y) = a_1 + a_2 x + a_3 y + a_2 (x + a_3) \ln (x + a_3)$, where a is a constant.

Proof.

(i) Consider these Taylor's expansions

$$f(x) = f(c) + f'(c)(x - c) + R_1(x), c \neq 0, \tag{1}$$

$$f(x) = f(c) + R_2(x), \tag{2}$$

where R is the remainder and c is a constant. Taking the derivatives of the remainders w.r.t. x yields

$$R_1'(x) = f'(x) - f'(c),$$

$$R_2'(x) = f'(x).$$

Thus

$$R_2'(x) - R_1'(x) = f'(c). \tag{3}$$

Dividing both sides of (3) by $x - u + \alpha$ and using the mean value theorem, we obtain

$$R_1''(x) = \frac{R_2'(x) - R_1'(x)}{x - u + \alpha} = \frac{f'(c)}{x - u + \alpha}, c \leq u \leq x, x - u + \alpha \neq 0.$$

where α is a positive constant. Integrating the above equation yields

$$\begin{aligned} R_1(x) &= \int_c^x \int_c^x \frac{f'(c)}{x-u+\alpha} dudu = \\ & f'(c)\left[\alpha \ln(\alpha) + x - ((x-c+\alpha)\ln(x-c+\alpha)+c)\right]. \end{aligned} \tag{4}$$

Substituting (4) into (1), we obtain

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \\ & f'(c)\left[\alpha \ln(\alpha) + x - ((x-c+\alpha)\ln(x-c+\alpha)+c)\right]. \end{aligned}$$

We can rewrite the above equation as

$$f(x) = a_1 + a_2 x + a_2 (x + a_3) \ln(x + a_3), \tag{5}$$

where a is a constant.□

(ii) As before we consider these Taylor's expansions

$$f(x,y) = f(c_1,c_2) + f_x(c_1,c_2)(x-c_1) + f_y(c_1,c_2)(y-c_2) + R_1(x,y), \tag{6}$$

$$f(x,y) = f(c_1,c_2) + R_2(x,y). \tag{7}$$

Taking the partial derivatives of the remainders w.r.t. x yields

$$R_{1x}(x,y) = f_x(x,y) - f_x(c_1,c_2),$$

$$R_{2x}(x,y) = f_x(x,y).$$

Therefore

$$R_{2x}(x,y) - R_{1x}(x,y) = f_x(c_1,c_2).$$

Thus

$$R_{1xx}(.) = \frac{f_x(c_1,c_2)}{x-u+\alpha}.$$

Integrating yields

$$R_1(x,y) = \int_{c_1}^{x}\int_{c_1}^{x} \frac{f_x(c_1,c_2)}{x-u+\alpha}dxdx =$$

$$f_x(c_1,c_2)\left[\alpha\ln(\alpha) + x - ((x-c_1+\alpha)\ln(x-c_1+\alpha)+c_1)\right]. \tag{8}$$

Substituting (8) into (6) yields

$$f(x,y) = a_1 + a_2x + a_3y + a_2(x+a_3)\ln(x+a_3).\Box \tag{9}$$

The extension to a multiple-variable function is straightforward.

3 Practical examples

In this section, we demonstrate the revolutionary nature of this contribution. In particular, we use it to transform sophisticated PDEs to non-differential equations or first-order ODEs.

The reaction-convection-diffusion equation:

The reaction-convection-diffusion equation has many applications in operations research, physics and finance. An example of such an equation is

$$V_t + rxV_x + \frac{1}{2}\sigma^2 x^2 V_{xx} - rV = 0, \tag{10}$$

subject to a boundary condition. Using (9) yields

$$V(x,t) = a_1 + a_2 x + a_3 t + a_2 (x + a_3) \ln (x + a_3). \tag{11}$$

Taking the partial derivatives of (11) yields

$$V_x(x,t) = a_2 [2 + \ln (x + a_3)], \tag{12}$$

$$V_t(x,t) = a_3, \tag{13}$$

$$V_{xx}(x,t) = \frac{a_2}{x + a_3}. \tag{14}$$

Substituting (12) $-$ (14) into (10), we obtain

$$a_3 + rxa_2 \left[2 + \ln \left(x + a_3\right)\right] + \frac{1}{2} \frac{a_2 \sigma^2 x^2}{x + a_3} - rV = 0.$$

Needless to say, we transformed (10) to a non-differential equation, such that V is given explicitly.

The heat equation:

An example of a diffusion equation is the heat equation

$$V_t - kV_{xx} = 0. \tag{15}$$

As before, substituting (14) into (15) yields

$$V_t - \frac{ka_2}{x + a_3} = 0.$$

Clearly, we transformed the heat equation to an ODE. Moreover, V is continuously differentiable and thus, a unique classical solution exists. Also, this method is easily applicable to non-linear PDEs.

The portfolio model:

We first provide a brief description of the baseline portfolio model (for an excellent review, see Kolm et al (2014)). Our approach may also be applicable to other variants of this model (see, for example, Palczewski et al (2015) and Alghalith (2012), among others). The risk-free asset price process is given by $S_0 = e^{\int_t^T r_s ds}$, where $r_t \in C_b^2 \left(R\right)$ is the risk-free rate of return. The dynamics of

the risky asset price are given by

$$dS_s = S_s\left(\mu_s ds + \sigma_s dW_s\right), \tag{16}$$

where $\mu_s \in C_b^2(R)$ and $\sigma_s \in C_b^2(R)$ are the rate of return and the volatility, respectively; W_s is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_s, P)$, where $\{\mathcal{F}_s\}_{t\leq s\leq T}$ is the augmentation of filtration.

The wealth process is given by

$$X_T^\pi = x + \int_t^T \left\{rX_s^\pi + \left(\mu_s - r_s\right)\pi_s\right\} ds + \int_t^T \pi_s\sigma_s dW_s, \tag{17}$$

where x is the initial wealth, $\{\pi_s, \mathcal{F}_s\}_{t\leq s\leq T}$ is the portfolio process, with $E\int_t^T \pi_s^2 ds < \infty$. The trading strategy $\pi_s \in \mathcal{A}(x, y)$ is admissible.

The investor maximizes the expected utility of the terminal wealth

$$V(t, x) = \underset{\pi}{Sup} E\left[U\left(X_T^\pi\right) \mid \mathcal{F}_t\right],$$

where $V(.)$ is the value function, $U(.)$ is a continuous, bounded and strictly concave utility function. Under well-known assumptions, the value function satisfies the Hamilton-Jacobi-Bellman PDE

$$V_t + rxV_x + \pi_t^*\left(\mu_t - r_t\right)V_x +$$

$$\frac{1}{2}\pi_t^{*2}\sigma_t^2 V_{xx} = 0; V(T,x) = U(x).$$

Following the procedure in the previous examples, we can rewrite the above Hamilton-Jacobi-Bellman PDE as

$$V_t + a_2\left[rx + \pi_t^*(\mu_t - r_t)\right]\left[2 + \ln(x + a_3)\right] + \frac{1}{2}\frac{a_2\pi_t^{*2}\sigma_t^2}{x + a_3} = 0,$$

which is clearly a first-order ODE. The optimal portfolio is given by

$$\pi_t^* = -\frac{(\mu_t - r_t)V_x}{\sigma_t^2 V_{xx}} = -\frac{a_2(\mu_t - r_t)\left[2 + \ln(x + a_3)\right]}{\sigma_t^2\left(\frac{a_2}{x+a_3}\right)}.$$

Thus, we explicitly expressed the optimal portfolio as a function of the initial wealth, even if the utility function is unknown.

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